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Polynomials in Many Variables: Real vs Complex Norms

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We study, for polynomials in many variables, the relations between the complex and the real sup-norms, and we give estimates involving the *leading coefficients*. We consider the case when the polynomial has a given degree, or some concentration at a given degree. The present paper is a contribution to a general field of investigation: For polynomials in many variables, what are the estimates independent of the number of variables?

In the sequel, we let

$$P(x_1, ..., x_N) = \sum_{|\alpha| \leqslant k} a_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N}, \tag{1}$$

with $\alpha = (\alpha_1, ..., \alpha_N)$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, be a polynomial of total degree at most k, in several variables $(x_1, ..., x_N)$. We consider here two problems related to the sup-norm of the polynomial: comparing the sup-norms in the

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real and in the complex cases, and finding bounds from below for this sup-norm when only certain coefficients of the polynomial are known, again in the real and in the complex cases.

We are interested here in estimates independent of the number of variables, as was already the case in Beauzamy, Bombieri, Enflo, and Montgomery [3]. As we will see, the complex and real cases require very different techniques, the former being much easier. Also, the estimates will sometimes be different, depending on whether the polynomial is homogeneous or not.

We start with the second problem, which might be called "reducing the polynomial," for we ask whether the knowledge of certain important terms (the *leading ones*) is enough to get a lower estimate on the sup-norm of the whole polynomial.

1. Reducing the Polynomial

Let P be written as in (1). We call *leading terms* those which contain only one variable, raised to the power k. Denoting by a_l, $l = 1, ..., n$, the leading coefficients, we write the polynomial as

$$P(x_1, ..., x_N) = \sum_1^N a_l x_l^k + \sum_{|\beta| \leq k} a_\beta x_1^{\beta_1} \cdots x_N^{\beta_N}, \tag{2}$$

where in the last terms all β's with $|\beta| = k$ have at least two non-zero components.

Our concern in this section is the following: If we know only the quantity $\sum_1^N |a_l|$, with no information at all on the rest of the polynomial, is this enough to get some information about its sup-norm? To distinguish in this way some terms which are of special importance is an idea frequently met in Partial Differential Equations. For instance, the Sobolev norm in the space $H^k(\Omega)$ is defined as

$$\|f\| = \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|_2 \right)^{1/2},$$

where $\|\cdot\|_2$ is the usual L_2 norm, and $D^\alpha = D_1^{\alpha_1} \cdots D_N^{\alpha_N}$, $D_j = \partial/\partial x_j$. When the domain is bounded, the above norm is equivalent to the norm obtained by taking only $\sum_{|\alpha| = k}$: the number of terms has been reduced.

The same idea holds for a differential operator, with constant coefficients, on a bounded domain. We define

$$P^{(\alpha)}(x_1, ..., x_N) = D_1^{\alpha_1} \cdots D_N^{\alpha_N} P(x_1, ..., x_N).$$

Then Hörmander's inequality states that, for any α, there is a positive constant C_α such that for any function ϕ in the test class C_0^∞,

$$\|P^{(\alpha)}(D)\,\phi\|_2 \leqslant C_\alpha \|P(D)\,\phi\|_2,$$

which, once again, allows one to reduce the number of terms in the differential operator.

Regarding the leading coefficients, we have, in the complex case:

THEOREM 1.1. *Let P be a polynomial of total degree k, with complex coefficients, written as in* (2). *Then*

$$\sum_1^N |a_l| \leqslant \max_{\theta_1, ..., \theta_N \in [0, 2\pi]} |P(e^{i\theta_1}, ..., e^{i\theta_N})|.$$

This theorem was proved, in the case of homogeneous polynomials, by Aron and Globevnik [1], using the multilinear form associated to the polynomial. The proof we present now is quite elementary, and is valid also for non-homogeneous polynomials.

Proof. Let A be the constant term in the polynomial. We may of course assume that this coefficient is real and positive.

For $l = 1, ..., N$, we write the leading coefficients in the form $a_l = \rho_l e^{i\theta_l}$. Let now $X_1, ..., X_N$ be independent random variables, defined on a probability space Ω, with values on the unit circle. More precisely, each X_l takes the k values $e^{-i(\theta_l + 2\pi m)/k}$, for $m = 0, ..., k-1$. Each value is taken with probability $1/k$. Let $\mathbf{E}$ be the expectation. We have

$$\mathbf{E}(X_l^k) = e^{-i\theta_l}, \qquad \mathbf{E}(a_l X_l^k) = |a_l|,$$

and $\mathbf{E}(X_l^j) = 0$ if $j < k$. We now consider the probability space $\Omega_1 \times \cdots \times \Omega_N$; let $\mathbf{E}_j$ be the expectation on Ω_j. We get

$$\mathbf{E}_1 \cdots \mathbf{E}_N P(X_1, ..., X_N) = \sum_1^N |a_l| + A \geqslant \sum_1^N |a_l|,$$

which shows that

$$\sum_1^N |a_l| \leqslant \max_{\theta_1, ..., \theta_N \in [0, 2\pi]} |P(e^{i\theta_1}, ..., e^{i\theta_N})|,$$

as we announced.

We remark that even more is true, in the complex case. Namely, let's write the polynomial P in Theorem 1.1 as $P = P_0 + P_1 + \cdots + P_k$, where each P_j is j-homogeneous. Further, write

$$P_j(x_1, ..., x_N) = \sum_1^N b_{jl} x_l^j + \sum_{|\beta| = j} b_\beta x_1^{\beta_1}, ..., x_N^{\beta_N},$$

where each β has at least two non-zero components. Applying Theorem 1.1 to P_j, we get

$$\sum_{l=1}^N |b_{jl}| \leqslant \max_{\theta_1, ..., \theta_N \in [0, 2\pi]} |P_j(e^{i\theta_1}, ..., e^{i\theta_N})| \leqslant \max_{\theta_1, ..., \theta_N \in [0, 2\pi]} |P(e^{i\theta_1}, ..., e^{i\theta_N})|,$$

by Cauchy's inequality.

We now turn to the case of real coefficients and real variables. The techniques will be quite different, and, as we will see, a result independent of the degree cannot hold anymore. We start with homogeneous polynomials.

THEOREM 1.2. *Let P be a homogeneous polynomial of degree k, written as in* (2). *Then*

$$\sum_1^N |a_l| \leqslant 4k^2 \max_{x_1, ..., x_N \in [0, 1]} |P(x_1, ..., x_N)|.$$

Proof. First we choose a subset L of $\{1, ..., N\}$ such that all the a_l's, $l \in L$, have the same sign and satisfy

$$\left|\sum_{l \in L} a_l\right| \geqslant \frac{1}{2} \sum_1^N |a_l|. \tag{3}$$

We may assume that $L = \{1, ..., m\}$, for some $m \leqslant N$, and that $a_l > 0$ for $l \leqslant m$. We give the value 0 to all variables x_l for $l > m$, and we just write $P(x_1, ..., x_m)$ instead of $P(x_1, ..., x_N)$. Finally, we normalize P, taking $\sum_1^m a_l = 1$.

Let now $X_1, ..., X_m$ be independent random variables, with values in $\{0, 1\}$, and with the same law: they take the value 1 with probability t, the value 0 with probability $1 - t$, where $t \in [0, 1]$ will be chosen later.

For every α, every l, $\mathbf{E} X_l^{\alpha_l} = \mathbf{E} X_l = t$. Let us denote by $c(\alpha)$ the number of non-zero components in α, and define

$$A_2 = \sum_{c(\alpha) = 2} a_\alpha, ..., A_k = \sum_{c(\alpha) = k} a_\alpha.$$

Then we get

$$\mathbf{E}_1 \cdots \mathbf{E}_m P(X_1, ..., X_m) = t + A_2 t^2 + \cdots + A_k t^k.$$

Let's call $S(t)$ the above polynomial, and put $\tilde{S}(x) = S((x+1)/2)$, for $-1 \leqslant x \leqslant 1$. Using Markov's inequality (see for instance Rivlin [10, p. 105]), we get

$$\begin{aligned} \max_{t \in [0,1]} |S(t)| &= \max_{x \in [-1,1]} |\tilde{S}(x)| \\ &\geqslant \frac{1}{k^2} \max_{x \in [-1,1]} |\tilde{S}'(x)| \\ &\geqslant \frac{1}{2k^2} \max_{t \in [0,1]} |S'(t)| \\ &\geqslant \frac{1}{2k^2}. \end{aligned}$$

This shows that there is a choice of values $x_1, ..., x_m$ in $\{0, 1\}$ such that

$$|P(x_1, ..., x_m)| \geqslant \frac{1}{2k^2},$$

and taking (3) into account, we finally obtain

$$|P(x_1, ..., x_m)| \geqslant \frac{1}{4k^2} \sum_1^N |a_l|,$$

as we announced.

Remark. We observe that homogeneity is not really necessary. The key to the proof is the fact that the polynomial $S(t)$ should start with t. This holds as soon as the polynomial P, written in canonical form (2), contains no simple terms, except the leading ones. Precisely, the same proof works if, in P, all coefficients a_β are 0 if $c(\beta) = 1$, $|\beta| \leqslant k-1$.

How good is the estimate given by Theorem 1.2? Let C_k be the best constant, that is, the smallest constant such that the inequality

$$\sum_1^N |a_l| \leqslant C_k \max_{x_i \in [-1,1]} |P(x_1, ..., x_N)|$$

holds for all homogeneous polynomials of degree k, in many variables. We now give an example, showing that $C_k \geqslant k$, at least when k is a power of 2.

EXAMPLE 1.3. We define by induction a sequence of polynomials by $p_1(x_1, x_2) = x_1^2 - x_2^2$. Assume $p_n(x_1, ..., x_{2^n})$ has been defined. Then put

$$p_{n+1}(x_1, ..., x_{2^{n+1}}) = p_n(x_1, ..., x_{2^n})^2 - p_n(x_{2^n+1}, ..., x_{2^{n+1}})^2.$$

Then, for every n, p_n is a polynomial of degree 2^n, with 2^n variables, and

$$|p_n(x_1, ..., x_{2^n})| \leqslant 1, \qquad \text{when} \quad x_1, ..., x_n \in [-1, 1].$$

The sum of leading coefficients is 2^n, which shows that $C_{2^n} \geqslant 2^n$.

Therefore we have the estimate

$$k \leqslant C_k \leqslant 4k^2.$$

We do not know the exact order of magnitude of C_k. There is an important case, however, in which C_k is proportional to k:

THEOREM 1.4. *Let P be a homogeneous polynomial of degree k, with real coefficients and real variables, written in canonical form* (2). *Assume that all coefficients a_β have the same sign. Then*

$$\sum_1^N |a_l| \leqslant D_k \max_{x_1, ..., x_N \in [0,1]} |P(x_1, ..., x_N)|,$$

where

$$D_k = \inf_{\lambda > 0} \frac{2(\lambda k + 1)((\lambda + 1)(\lambda k - \lambda + 1) + 1)}{\lambda^2(k-1)},$$

thus satisfying

$$D_k \leqslant \frac{2(k+1)(2k+1)}{k-1}; \qquad \limsup_{k \to \infty} \frac{D_k}{k} \leqslant 2.$$

Proof. First, we make the same reductions as in Theorem 1.2: we assume that for $l = 1, ..., m$ the a_l's are positive and satisfy $\sum_1^m a_l = 1$. Since the proof is trivial if all other coefficients are also positive, we assume them to be negative, and we write

$$P(x_1, ..., x_m) = \sum_1^m a_l x_l^k - \sum_\beta a_\beta x_1^{\beta_1} \cdots x_m^{\beta_m}, \tag{4}$$

where all the a_β's are positive.

Fix now $\lambda > 0$ to be chosen later, and put

$$\varepsilon(\lambda) = \frac{(k-1)\lambda^2}{(k^2-k)\lambda^3 + (k^2+k-1)\lambda^2 + 3k\lambda + 2}. \tag{5}$$

We now have two cases:

— if $|P(1, ..., 1)| \geqslant \varepsilon(\lambda)$, our result is proved,

— otherwise, it means that

$$\left|1 - \sum a_\beta\right| \leqslant \varepsilon(\lambda). \tag{6}$$

We need a lemma:

LEMMA 1.5. *Fix* $\lambda > 0$. *Under the constraints* $\beta_1 + \cdots + \beta_m = k$, $\max \beta_j < k$, *one has*

$$(\lambda\beta_1 + 1) \cdots (\lambda\beta_m + 1) \geqslant (\lambda + 1)(\lambda k - \lambda + 1).$$

Proof of Lemma 1.5. The minimum of the function $(\lambda x + 1)(\lambda y + 1)$ under the constraints $x + y = k$, $0 \leqslant x \leqslant k$, $0 \leqslant y \leqslant 1$, is reached at $x = 0$, $y = k$, or at $x = k$, $y = 0$, and is $\lambda k + 1$. This shows that

$$(\lambda\beta_1 + 1)(\lambda\beta_2 + 1) \geqslant \lambda\beta_1 + \lambda\beta_2 + 1,$$

and so on with subsequent terms.

But the minimum of $(\lambda x + 1)(\lambda y + 1)$ under the constraints $x + y = k$, $0 \leqslant x \leqslant k-1$, $0 \leqslant y \leqslant k-1$, is reached at $x = 0$, $y = k-1$, or at $x = k-1$, $y = 0$, and its value is $(\lambda + 1)(\lambda k - \lambda + 1)$. Since in each term at least two of the β_j's are non-zero, the lemma follows.

We now come back to the proof of the theorem. By (6) and Lemma 1.5, we have

$$\begin{aligned}
\int_0^1 \cdots \int_0^1 & P(x_1^\lambda, ..., x_m^\lambda)\, dx_1 \cdots dx_m \\
&= \frac{1}{\lambda k + 1} - \sum a_\beta \frac{1}{(\lambda\beta_1 + 1)\cdots(\lambda\beta_m + 1)} \\
&\geqslant \frac{1}{\lambda k + 1} - \frac{1 + \varepsilon(\lambda)}{(\lambda + 1)(\lambda k - \lambda + 1)} \\
&\geqslant \varepsilon(\lambda),
\end{aligned}$$

by the choice of $\varepsilon(\lambda)$.

This shows that, in this case also, for every $\lambda > 0$, there are points $x_1, ..., x_m$ in $[0, 1]$ such that

$$|P(x_1, ..., x_m)| \geqslant \varepsilon(\lambda) \geqslant \frac{\varepsilon(\lambda)}{2} \sum_1^N |a_l|,$$

from which the theorem follows, taking the maximum of $\varepsilon(\lambda)$ over all $\lambda > 0$.

We now turn to polynomials with real coefficients, but which are not necessarily homogeneous. There is a considerable quantitative difference from the homogeneous case: the constant C_k must be exponential. Indeed, this is already the case with polynomials in one variable: consider the Tchebycheff polynomial

$$T_k(x) = \prod_1^k \left(x - \cos \frac{(2j+1)\pi}{2k} \right),$$

normalized so as to have leading coefficient 1. Then

$$\max_{x \in [0,1]} |T_k(x)| = \frac{1}{2^{k-1}}.$$

In fact, dealing now with many variables, we will lose exactly the same factor $1/4k^2$ as we already did in the homogeneous case.

THEOREM 1.6. *Let P be a polynomial of total degree k, in many variables, with real coefficients, written in canonical form* (2). *Then*

$$\sum_1^N |a_l| \leqslant 2^{k+2}k^2 \max_{x_1, ..., x_N \in [-1,1]} |P(x_1, ..., x_N)|.$$

Proof. Let A be, as before, the constant coefficient $P(0, ..., 0)$. We have to distinguish between two cases, depending on the size of $|A|$. First, if

$$|A| \geqslant \frac{1}{2^{k+2}k^2} \sum_1^N |a_l|,$$

our estimate is proved. So we now assume the converse inequality. We make the same reductions as in Theorem 1.2 above, and, assuming $\sum_1^m a_l = 1$, we get

$$|A| \leqslant \frac{1}{2^{k+1}k^2} \sum_1^m a_l. \tag{7}$$

Fix now $s \in [-1, 1]$, and for $l = 1, ..., m$, consider independent random variables X_l, with values in the set $\{0, s\}$, and all with the same law: the probability of the value 0 is $1 - t$, the probability of the value s is t. Then the contributions of the parts of P of respective degrees $k, k-1, ..., 0$, give

$$\begin{aligned}
&\mathbf{E}_1 \cdots \mathbf{E}_m P(X_1, ..., X_m) \\
&\quad = s^k t + s^k (A_{2,k} t^2 + \cdots + A_{k,k} t^k) \\
&\quad\quad + s^{k-1} (A_{1,k-1} t + \cdots + A_{k-1,k-1} t^{k-1}) \\
&\quad\quad \vdots \\
&\quad\quad + s^j (A_{1,j} t + \cdots + A_{j,j} t^{k-j}) \\
&\quad\quad \vdots \\
&\quad\quad + A,
\end{aligned}$$

with

$$A_{u,v} = \sum_{\substack{|\beta| = v \\ c(\beta) = u}} a_\beta.$$

The coefficient of t is

$$f(s) = s^k + A_{1,k-1} s^{k-1} + \cdots + A_{1,j} s^j + \cdots + A_{1,0},$$

which is a polynomial in s of degree k, with leading coefficient 1.

For such a polynomial, there is always (see Rivlin [10, p. 67]) a point $s \in [-1, 1]$ such that

$$|f(s)| \geqslant \frac{1}{2^{k-1}}.$$

Fix this s, and put $\lambda = f(s)$. Then $\mathbf{E}_1 \cdots \mathbf{E}_m P(X_1, ..., X_m)$ is now a polynomial in t, of the form

$$\mathbf{E}_1 \cdots \mathbf{E}_m P(X_1, ..., X_m) = \lambda(t + B_2 t^2 + \cdots + B_k t^k) + A.$$

Reasoning as we did in Theorem 1.2, we find that there is a value of t such that

$$|t + B_2 t^2 + \cdots + B_k t^k| \geqslant \frac{1}{2k^2},$$

which shows that there are points $(x_1, ..., x_m)$ in $[-1, 1]$ for which

$$|P(x_1, ..., x_m)| \geqslant \frac{1}{k^2 2^k} - A \geqslant \frac{1}{k^2 2^{k+1}} \geqslant \frac{1}{k^2 2^{k+2}} \sum_1^N |a_l|,$$

which proves the theorem.

Remark 1.7. One can wonder if a result similar to those of Theorems 1.1 to 1.6 holds if one replaces the sum of leading terms, $\sum_1^N |a_i|$, by the sum of all terms, $\sum_\alpha |a_\alpha|$, that is, the l_1-norm of the polynomial, written as in (1). Namely, is there a constant C'_k, independent of the number of variables, such that

$$\sum_\alpha |a_\alpha| \leqslant C'_k \|P\|_\infty, \tag{8}$$

where $\|\cdot\|_\infty$ is any of the sup-norms we have considerd before? Such a result would imply (at least qualitatively) all the previous ones. In the one-variable case, it holds with $C'_k = \sqrt{k+1}$.

In the many-variables case, it holds for $k = 1$ (with $C'_1 = 1$), but fails for any $k \geqslant 2$: this was known to Littlewood [9], as a consequence of his theory on continuous bilinear forms on $c_0 \times c_0$. Another proof is as follows.

By a result of Bennett, Goodman, and Newman [4], there is a $N \times N$ matrix $A = (a_{i,j})$, with entries ± 1, such that

$$\|A\|_{op} \sim \sqrt{N},$$

where $\|A\|_{op}$ is the operator norm of A from $l_2^{(N)}$ into itself, that is,

$$\|A\|_{op} = \max\{\|AX\|_2;\ \|X\|_2 = 1\},$$

with

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \qquad \|X\|_2 = \left(\sum_1^N |x_j|^2\right)^{1/2}.$$

Set $P_0(x_1, ..., x_N) = \sum_{i,j=1}^N a_{i,j} x_i x_j$. It has degree 2, and with X as before,

$$P_0(x_1, ..., x_N) = \langle AX, X \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in l_2.

So we get

$$|P_0(x_1, ..., x_N)| \leqslant \|AX\|_2 \|X\|_2 \leqslant c\sqrt{N}\, \|X\|_2^2,$$

where c is some absolute constant. But if $|x_j| \leqslant 1$ for $j = 1, ..., N$, then $\|X\|_2 \leqslant \sqrt{N}$, and

$$|P_0(x_1, ..., x_N)| \leqslant cN^{3/2},$$

a contradiction with (8), since $|P_0|_1 = \sum_{i,j} |a_{i,j}| = N^2$.

Remark 1.8. In Beauzamy, Bombieri, Enflo, and Montgomery [3], Lemma 1.A.3 shows that

$$|P|_1 \sim \sum_1^N \left|\frac{\partial P}{\partial x_i}\right|_1,$$

with constants independent of the number of variables. Of course, from Euler's formula follows

$$\|P\|_\infty \leqslant \sum_1^N \left\|\frac{\partial P}{\partial x_i}\right\|_\infty,$$

but the same polynomial as before shows that the converse inequality, for the L_∞-norm, is not possible with a constant independent of the number of variables, since

$$\sum_1^N \left\|\frac{\partial P}{\partial x_i}\right\|_\infty = N^2; \qquad \|P\|_\infty \leqslant cN^{3/2}.$$

We now turn to the comparison between the sup-norm in the real and in the complex cases.

2. Comparison between Sup-norms in the Real and in the Complex Cases

Let $P(x_1, ..., x_N)$ be as before a polynomial of degree k, in many variables, with real coefficients, written as in (1). We want to compare the two quantities

$$\|P\|_{\infty, R} = \max_{x_1, ..., x_N \in [-1, 1]} |P(x_1, ..., x_N)|,$$

$$\|P\|_{\infty, C} = \max_{|z_1| = \cdots = |z_N| = 1} |P(z_1, ..., z_N)|;$$

the latter being obviously larger than the former, we are interested in estimates in the other direction. We obtain:

THEOREM 2.1. *For any polynomial P of degree k, with real coefficients,*

$$\|P\|_{\infty, R} \geqslant \mu_k \|P\|_{\infty, C},$$

where $\mu_k = 2/((2+\sqrt{2})^k + (2-\sqrt{2})^k)$.

We first need a lemma:

LEMMA 2.2. *Let P be any polynomial of degree k, in n variables. Then, for any* $\lambda \in [0, 1]$,

$$\max_{|z_1| = \cdots = |z_N| = \lambda} |P(z_1, ..., z_N)| \geq \lambda^k \max_{|z_1| = \cdots = |z_N| = 1} |P(z_1, ..., z_N)|.$$

Proof of Lemma 2.2. Assume $\max_{|z_j|=1} |P(z_1, ..., z_N)| = 1$. Let $z_1, ..., z_N$ be points on the unit circle such that $|P(z_1, ..., z_N)| = 1$. Consider $f(z) = P(zz_1, ..., zz_N)$, where z is a complex number. Then f is a polynomial in one complex variable, of degree k. By a classical result for polynomials in one variable (see for instance A. Durand [6, Theorem 3.7, p. 18]),

$$\max_{|z| = \lambda} |f(z)| \geq \lambda^k \max_{|z| = 1} |f(z)| \geq \lambda^k |f(1)| = \lambda^k,$$

which shows that there is a z, $|z| = \lambda$, such that

$$|P(zz_1, ..., zz_N)| \geq \lambda^k,$$

and proves our claim.

We now prove the theorem. We assume $\|P\|_{\infty, C} = 1$. Using Lemma 2.2, we find points $z_1, ..., z_N$, with $|z_j| = 1/\sqrt{2}$, such that

$$|P(z_1, ..., z_N)| \geq \left(\frac{1}{\sqrt{2}}\right)^k \|P\|_{\infty, C}.$$

We write $z_j = c_j + id_j$, where c_j and d_j are real. We now look at

$$\phi(t) = \sum_{|\alpha| \leq k} a_\alpha (c_1 + td_1)^{\alpha_1} \cdots (c_N + td_N)^{\alpha_N},$$

which is a polynomial of degree k in t, satisfying $\|\phi\|_\infty \geq 1/\sqrt{2}^k$.

A result of P. Erdős [7, Theorem 7, p. 1175, and Corollary, p. 1176] shows that if ϕ is a polynomial of degree k, with real coefficients,

$$\|\phi\|_{\infty, C} \leq |T_k(i)| \, \|\phi\|_{\infty, R}, \tag{9}$$

where T_k is the Tchebycheff polynomial of degree k. Therefore we get

$$\|\phi\|_{\infty, R} \geq \mu_k,$$

with

$$\mu_k = \frac{2}{(2 + \sqrt{2})^k + (2 - \sqrt{2})^k}.$$

So there exists a t in $[-1, 1]$ such that

$$\left|\sum_{|\alpha| \leqslant k} a_\alpha (c_1 + td_1)^{\alpha_1} \cdots (c_N + td_N)^{\alpha_N}\right| \geqslant \mu_k.$$

Since for every j, $c_j + td_j$ belongs to $[-1, 1]$, our theorem is proved.

For homogeneous polynomials, another proof can be given, using the multi-linear functional associated to the polynomial. It leads to estimates which are slightly weaker, but the proof is much simpler.

Let $P(z_1, ..., z_N)$ be an homogeneous polynomial of degree k. The associated k-linear form A, from $\mathbb{C}^n \times \cdots \times \mathbb{C}^n$ into $\mathbb{C}$ is defined by

$$A(Z^{(1)}, ..., Z^{(k)}) = \frac{1}{k!\,2^k} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_k P\left(\sum_{j=1}^{k} \varepsilon_j Z^{(j)}\right),$$

where each Z stands for $(z_1, ..., z_N)$. A proof of the fact that A is k-linear can be found for instance in Dineen [5]. No matter whether the variables are real or complex, the same computation shows that

$$\max_{Z^{(j)} \in \mathbb{C}^N, \|Z^{(j)}\| \leqslant 1} |A(Z^{(1)}, ..., Z^{(k)})| \leqslant \frac{k^k}{k!} \max_{|z_l| \leqslant 1} |P(z_1, ..., z_N)|,$$

$$\max_{X^{(j)} \in \mathbb{R}^N, \|X^{(j)}\| \leqslant 1} |A(X^{(1)}, ..., X^{(k)}| \leqslant \frac{k^k}{k!} \max_{|x_l| \leqslant 1} |P(x_1, ..., x_N)|,$$

where $\|Z\| = (\sum_1^N |z_l|^2)^{1/2}$.

Now, we get, writing $Z = X + iY$ in $\mathbb{C}^N$,

$$\begin{aligned}\max_{|z_l| \leqslant 1} |P(z_1, ..., z_N)| &\leqslant \max_{\|Z^{(j)}\| \leqslant 1} |A(Z^{(1)}, ..., Z^{(k)})| \\ &\leqslant 2^k \max_{\|X^{(j)}\| \leqslant 1} |A(X^{(1)}, ..., X^{(k)})| \\ &\leqslant \frac{2^k k^k}{k!} \max_{|x_l| \leqslant 1} |P(x_1, ..., x_N)|,\end{aligned}$$

which gives a constant $(2k)^k/k!$, slightly worse than that of the previous proof.

Remark 2.3. Theorem 2.1 was stated only for polynomials with real coefficients, because its proof uses Erdős' estimate (9), which was proved only for such polynomials. The general question of a sharp bound

$$\|\phi\|_{\infty, C} \leqslant C_k \|\phi\|_{\infty, R},$$

valid for all polynomials of degree k, with complex coefficients, remains open. An interesting advance in this direction has been made by R. Freund and S. Ruscheweyh [8].

Remark 2.4. In the one-variable case, the equivalence between the complex and real sup-norms is obtained considering the quantity

$$\max_{0 \leqslant \delta \leqslant 1} \min_{|z| = \delta} |P(z)|,$$

which is smaller than the norm $\|P\|_{\infty, R}$, and is shown to be equivalent to the norm $\|P\|_{\infty, C}$ (see A. Durand [6, p. 20, Theorem 3.10]). This stronger equivalence cannot hold in the many-variable case.

Consider indeed the simple polynomial $P(z_1, ..., z_N) = z_1 + \cdots + z_N$. Then $\|P\|_{\infty, C} = N$, and

$$\begin{aligned}
&\max_{a_1, ..., a_N \in [0,1]} \min_{\theta_1, ..., \theta_N} |a_1 e^{i\theta_1} + \cdots + a_N e^{i\theta_N}| \\
&\leqslant \max_{a_1, ..., a_N \in [0,1]} \left(\int_0^{2\pi} \cdots \int_0^{2\pi} |a_1 e^{i\theta_1} + \cdots + a_N e^{i\theta_N}|^2 \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi} \right)^{1/2} \\
&\leqslant \max_{a_1, ..., a_N \in [0,1]} (a_1^2 + \cdots + a_N^2)^{1/2} \\
&= \sqrt{N},
\end{aligned}$$

which proves our claim.

3. Real vs Complex Sup-norms for Polynomials with Concentration at Low Degrees

Recall that a polynomial P, with N variables, of degree n, written in canonical form

$$P(x_1, ..., x_N) = \sum_{|\alpha| \leqslant n} a_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N} \tag{10}$$

has concentration d $(0 < d \leqslant 1)$ at degree k if

$$\sum_{|\alpha| \leqslant k} |a_\alpha| \geqslant d \sum_{|\alpha| \leqslant n} |a_\alpha|. \tag{11}$$

In the previous section, the estimates we gave involved the degree of the polynomial. In the present section, we consider the following question: Can we obtain similar estimates for the ratio between the complex and the real sup-norms if we do not take into account the degree itself, but only the *concentration at low degrees*, that is, the data d and k? We still want, of course, estimates independent of the number of variables. The reader may consult *Beauzamy, Bombieri, Enflo, and Montgomery [3] for related* definitions and results.

For the present problem, the answer will be that everything depends on the way this concentration is measured. So before giving the main theorem of this section, we give an example showing that the concentration cannot be measured the usual way, that is, using l_1-norm, as defined in (11).

EXAMPLE 3.1. Consider for every $N \geqslant 1$ the family of polynomials

$$P_N(z, z_1, ..., z_N) = \frac{1}{N^2} P_0(z_1, ..., z_N) + \frac{1}{2^N} (z^2 - 1)^N,$$

where P_0 is the degree 2 polynomial introduced in Remark 1.7; it has coefficients ± 1. Therefore

$$\left| \frac{1}{N^2} P_0 \right|_1 = 1,$$

and since $|(1/2^N)(z^2 - 1)^N| = 1$, the polynomial P_N has concentration 1/2 at degree 2.

We have seen in Remark 1.7 that

$$\left\| \frac{1}{N^2} P_0 \right\|_{\infty, C} \leqslant \frac{1}{\sqrt{N}}.$$

Therefore, giving to z the value i, we see that

$$\max_{|z|, |z_j| = 1} |P_N(z, z_1, ..., z_N)| \geqslant \max_{|z_j| = 1} |P_N(i, z_1, ..., z_N)| \geqslant 1 - \frac{1}{\sqrt{N}},$$

though

$$\max_{x, x_j \in [-1, 1]} |P_N(x, x_1, ..., x_N)| \leqslant \frac{1}{\sqrt{N}} + \frac{1}{2^N},$$

which shows that a result independent of N cannot exist in this case.

Such a statement will be possible if, instead of the l_1-norm, we use the L_∞-norm to measure the concentrations. With d, k as before, with P written as in (10), define

$$P'_k = \sum_{|\alpha| \leqslant k} a_\alpha z_1^{\alpha_1} \cdots z_N^{\alpha_N},$$

which is the part of P of degree at most k. Then we say that P has concentration d at degree k, measured in L_∞-norm, if

$$\|P'_k\|_{\infty,C} \geqslant d\|P\|_{\infty,C}. \tag{12}$$

Then we have:

THEOREM 3.2. *There is a constant $c(d,k)>0$ such that, for any polynomial P with real coefficients, having concentration d at degree k measured in L_∞-norm, one has*

$$\max_{x_j \in [-1,1]} |P(x_1, ..., x_N)| \geqslant c(d,k) \max_{|z_j|=1} |P(z_1, ..., z_N)|.$$

The constant $c(d,k)$ is independent of the degree and of the number of variables.

Proof. As usual, we normalize P in order to have $\|P\|_{\infty,C}=1$. So we get from (12)

$$\|P'_k\|_{\infty,C} \geqslant d.$$

By Theorem 2.1,

$$\|P'_k\|_{\infty,R} \geqslant d\mu_k.$$

This means that there are values $x_1, ..., x_N$ in $[-1,1]$ such that

$$|P'_k(x_1, ..., x_N)| \geqslant d\mu_k. \tag{13}$$

We now consider

$$f(z) = P(zx_1, ..., zx_N),$$

which is a polynomial of degree n in one complex variable z. Set

$$f(z) = \sum_0^n b_j z^j.$$

First we observe that

$$\|f\|_{\infty,C} \leqslant \|P\|_{\infty,C} = 1,$$

which implies

$$\left(\sum_0^n |b_j|^2\right)^{1/2} \leqslant 1. \tag{14}$$

Now from (13) it follows obviously that

$$\left|\sum_0^k b_j\right| \geqslant d\mu_k,$$

which implies

$$\left(\sum_0^k |b_j|^2\right)^{1/2} \geqslant \frac{d\mu_k}{\sqrt{k+1}}. \tag{15}$$

This, together with (14), shows that f has concentration $d' = d\mu_k/\sqrt{k+1}$ at degree k, measured in l_2-norm. By a result of Beauzamy [2], we know that for such a function, the set $\{z; |f(z)| < \varepsilon\}$ can be covered by a union of disks D_j, with radii $r(D_j)$ satisfying

$$\sum_j r(D_j) \leqslant \phi_{d,k}(\varepsilon),$$

where $\phi_{d,k}(\varepsilon)$ is a function depending only on the data d', k, which tends to 0 when $\varepsilon \to 0$; precisely

$$\phi_{d,k}(\varepsilon) = C\frac{\log\log 1/\sqrt{\varepsilon}}{\log 1/\sqrt{\varepsilon}} \log\frac{2^k}{d'},$$

where C is a universal constant.

Take now ε small enough to get

$$\phi_{d,k}(\varepsilon) < 2,$$

then the set $\{|f(z)| < \varepsilon\}$ cannot be contained in the segment $[-1, 1]$, which means that, for such a choice of ε,

$$\max_{t \in [-1,1]} |f(t)| \geqslant \varepsilon,$$

and proves our result.

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Note added in proof. Maciej Klimek (Dept. of Mathematics, University College, Dublin 4, Ireland) has recently obtained a significant improvement of Theorem 2.1. Using the Siciak extremal function (see e.g., [M. Klimek, *Pluripotential theory*, Oxford Univ. Press (1991)]), he has proved that for any complex polynomial P on C^n of degree k, $\|P\|_{\infty,C} \leqslant (1+\sqrt{2})^k \|P\|_{\infty,R}$.

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